MA 681, Spring 2013

Assignment 2.

Convergence in $\mathbb{C}$. The extended complex plane. Map $1/z$.

This assignment is due Wednesday, Jan 30. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Suppose the sequence (z_n) in $\mathbb{C}$ converges to $z \in \mathbb{C}$ as $n \to \infty$.
   (a) Is it true that $|z_n| \to |z|$, $\operatorname{Re} z_n \to \operatorname{Re} z$, $\operatorname{Im} z_n \to \operatorname{Im} z$ $(n \to \infty)$?
   (b) Is it true that $\arg z_n \to \arg z$ $(n \to \infty)$? (*Hint:* Consider $z = 0$.)
   (c) Is it true that $\arg z_n \to \arg z$ $(n \to \infty)$, provided $z \neq 0$? (*Hint:* Still no.)
   (d) Provided $z \neq 0$, is it possible to choose a value φ_n of $\operatorname{Arg} z_n$ for each n so that $\varphi_n \to \arg z$ $(n \to \infty)$?

(2) Suppose the sequence (z_n) in $\mathbb{C}$ converges to infinity as $n \to \infty$. What does this imply about $|z_n|$, $\operatorname{Re} z_n$, $\operatorname{Im} z_n$, $\operatorname{Arg} z_n$?

(3) Assuming arithmetic operations on $\overline{\mathbb{C}}$ are defined via arithmetic operations on the corresponding sequences, give examples showing why $\infty - \infty$, $0 \cdot \infty$, ∞/∞, $0/0$ are meaningless.

(4) Find and sketch the images of the following curves under the transformation $w = 1/z$ (*Hint:* It is probably more convenient to use complex equations for the families below):
   (a) The family of circles $x^2 + y^2 = ax$ $(a \in \mathbb{R})$. Remark: compare to Prob. 9 of HW1.
   (a') The family of vertical lines $\operatorname{Re} z = a$ $(a \in \mathbb{R})$.
   (b) The family of circles $x^2 + y^2 = by$ $(b \in \mathbb{R})$. Remark: compare to Prob. 9 of HW1.
   (b') The family of horizontal lines $\operatorname{Im} z = b$ $(b \in \mathbb{R})$.
   (c) The family of parallel lines $y = x + b$ $(b \in \mathbb{R})$.
   (d) The family of lines $y = kx$ passing through the origin $(k \in \mathbb{R})$.

(5) Find and sketch the images of the following regions on $\overline{\mathbb{C}}$ under the transformation $w = 1/z$ (*Hint:* Borders go to borders):
   (a) The quadrant $x > 0,\ y > 0$.
   (b) The strip $0 < x < 1$.
   (c) The half-plane $x > 10$.
   (d) The outside of a circle $x^2+(y-1)^2 = 1$ (i.e. the region $x^2+(y-1)^2 > 1$).
   (e) The outside of a circle $x^2+(y-2)^2 = 1$ (i.e. the region $x^2+(y-2)^2 > 1$).
   (f) The square $-1 \le x \le 1$, $-1 \le y \le 1$.
   (g) The square $-\sqrt{2}/2 \le x + y \le \sqrt{2}/2$, $-\sqrt{2}/2 \le x - y \le \sqrt{2}/2$.
   (h) The upper half-disc $|z| < 1$, $\operatorname{Im} z > 0$.